

Blowup behavior for a degenerate elliptic sinh-Poisson equation with variable intensities

Tonia Ricciardi* Ryo Takahashi †

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Abstract

In this paper, we provide a complete blow-up picture for solution sequences to an elliptic sinh-Poisson equation with variable intensities arising in the context of the statistical mechanics description of two-dimensional turbulence, as initiated by Onsager. The vortex intensities are described in terms of a probability measure \mathcal{P} defined on the interval $[-1, 1]$. Under Dirichlet boundary conditions we establish the exclusion of boundary blowup points, we show that the concentration mass does not have residual L^1 -terms (“residual vanishing”) and we determine the location of blowup points in terms of Kirchhoff’s Hamiltonian. We allow \mathcal{P} to be a general Borel measure, which could be “degenerate” in the sense that $\mathcal{P}(\{\alpha_-^*\}) = 0 = \mathcal{P}(\{\alpha_+^*\})$, where $\alpha_-^* = \min \text{supp} \mathcal{P}$ and $\alpha_+^* = \max \text{supp} \mathcal{P}$. Our main results are new for the standard sinh-Poisson equation as well.

1 Introduction

Since Onsager’s pioneering work [21] in 1949, the statistical mechanics description of stable, large scale vortices has attracted the attention of many physicists and mathematicians, and is still of central interest in fluid mechanics [5, 7]. In particular, several mean field equations have been proposed to describe two-dimensional stationary Euler flows with a large number of point vortices.

In this paper we are concerned with the following mean field equation derived by C. Neri in [18] under the “stochastic” assumption that the vortex intensities and orientations are independent identically distributed random variables with probability distribution $\mathcal{P}(d\alpha)$, $\alpha \in [-1, 1]$:

$$\begin{cases} -\Delta v = \lambda \int_{[-1,1]} \frac{\alpha e^{\alpha v}}{\iint_{[-1,1] \times \Omega} e^{\alpha' v} \mathcal{P}(d\alpha') dx} \mathcal{P}(d\alpha) & \text{in } \Omega \\ v = 0 & \text{on } \partial\Omega. \end{cases} \quad (1.1)$$

Here, v denotes the stream function of a turbulent Euler flow, $\Omega \subset \mathbf{R}^2$ is a bounded domain with smooth boundary $\partial\Omega$, $\lambda > 0$ is a constant related to

*Dipartimento di Matematica e Applicazioni “R. Caccioppoli” Università di Napoli Federico II, Via Cintia, 80126 Napoli, Italy (E-mail: tonia.ricciardi@unina.it)

†Division of Mathematical Science, Department of Systems Innovation, Graduate School of Engineering Science, Osaka University, Machikaneyamacho 1-3, Toyonakashi, 560-8531, Japan. (E-mail: r-takaha@sigmath.es.osaka-u.ac.jp)

the inverse temperature. We further assume that $\mathcal{P} \in \mathcal{M}([-1, 1])$ is a Borel probability measure defined on the interval $[-1, 1]$, where $\mathcal{M}([-1, 1])$ denotes the space of measures on $[-1, 1]$.

If $\mathcal{P} = \delta_{+1}$, corresponding to the case where all vortices have the same intensity and orientation, equation (1.1) reduces to the Liouville type equation

$$\begin{cases} -\Delta v = \lambda \frac{e^v}{\int_{\Omega} e^v dx} & \text{in } \Omega \\ v = 0 & \text{on } \partial\Omega \end{cases} \quad (1.2)$$

whose properties are by now well understood, see, e.g., [14, 31] and the references therein. If $\mathcal{P} = (\delta_{+1} + \delta_{-1})/2$, corresponding to the case where the point vortices have the same intensity and variable orientation, equation (1.1) reduces to the sinh-Poisson type problem:

$$\begin{cases} -\Delta v = \frac{\lambda}{2} \frac{e^v - e^{-v}}{\int_{\Omega} (e^v + e^{-v}) dx} & \text{in } \Omega \\ v = 0 & \text{on } \partial\Omega \end{cases} \quad (1.3)$$

Equation (1.3) is also related to the study of constant mean curvature surfaces and has received a considerable attention, see, e.g., [2, 12, 13, 29] and the references therein. Our results for (1.1) will yield new results for (1.3) as well.

It is useful to mention that another mean field equation with probability measures formally similar to (1.1) was derived under a “deterministic” assumption on the vortex intensities in [28], see also Onsager’s handwritten note in [10]:

$$\begin{cases} -\Delta v = \lambda \int_{[-1, 1]} \frac{\alpha e^{\alpha v}}{\int_{\Omega} e^{\alpha v} dx} \mathcal{P}(d\alpha) & \text{in } \Omega \\ v = 0 & \text{on } \partial\Omega. \end{cases} \quad (1.4)$$

Equation (1.4) also reduces to (1.2) when $\mathcal{P} = \delta_{+1}$. However, if $\mathcal{P} = (\delta_{+1} + \delta_{-1})/2$, equation (1.4) reduces to

$$\begin{cases} -\Delta v = \frac{\lambda}{2} \left(\frac{e^v}{\int_{\Omega} e^v dx} - \frac{e^{-v}}{\int_{\Omega} e^{-v} dx} \right) & \text{in } \Omega \\ v = 0 & \text{on } \partial\Omega \end{cases} \quad (1.5)$$

which is evidently different from (1.3). Results for equation (1.5) may be found in [9, 20] and the references therein.

It is therefore a natural question to seek common properties between (1.1) and (1.4) as well as different behaviors, which could in principle provide a criterion to select a “more suitable” model. Several results in this direction were obtained in [19, 23, 25, 26, 27]. In particular, with the aim of studying common properties of solution sequences on compact surfaces, a very general equation containing (1.1) and (1.4) as special cases was considered in [25]. Roughly speaking, it was shown that the basic Brezis-Merle type blow-up alternatives holds true for both models, see Proposition 1 below. On the other hand, consideration of the optimal Moser-Trudinger type inequalities associated to (1.1) and (1.4) emphasized significantly different properties between (1.1) and (1.4), see [23, 25, 30].

Our aim in this article is to complete the blow-up analysis for solution sequences to (1.1) initiated in [25, 26]. We shall first of all show that under Dirichlet boundary conditions blow-up cannot occur on the boundary. We notice that the exclusion of boundary blowup points (see Theorem 1-II-(i) for the precise statement) is not straightforward for general cases of \mathcal{P} , although it is readily derived from an estimate in [11] for the one-sided case, that is, the case where $\text{supp } \mathcal{P} \subset [0, 1]$ or $\text{supp } \mathcal{P} \subset [-1, 0]$. In a forthcoming paper [24] we shall show that the exclusion of boundary blow-up points holds true for a more general class of equations including (1.1) and (1.4) with Dirichlet boundary conditions. Then, we establish the vanishing of the L^1 -terms in the concentration mass limit (“residual vanishing”), see Theorem 1-II-(ii) or Section 4. This property was derived in [26] for \mathcal{P} satisfying $\text{supp } \mathcal{P} \subset [0, 1]$ and $\mathcal{P}(\{1\}) > 0$, but the case of a general Borel measure \mathcal{P} was left open. We note that the residual vanishing property is specific to (1.1), in the sense that it is known to be false for (1.4) for some special choices of \mathcal{P} , see [30]. As a consequence of the residual vanishing property we shall locate the blow-up points in terms of Kirchhoff’s Hamiltonian [3, 8, 31]

$$\mathcal{H}_N(x_1, \dots, x_N) = \sum_{i=1}^N r_i^2 H(x_i, x_i) + \sum_{\substack{i,j=1 \\ i \neq j}}^N r_i r_j G(x_i, x_j), \quad (1.6)$$

where $r_i \in [-1, 1]$, $i = 1, \dots, N$ denotes the vortex intensity of x_i and where the sign of r_i determines the vortex orientation. Here, $G = G(x, y)$ and $H = H(x, y)$ are the Green’s function and its regular part, respectively, that is,

$$-\Delta G(x, y) = \delta_y \text{ in } \Omega, \quad G(x, y) = 0 \text{ on } \partial\Omega \quad (1.7)$$

and

$$H(x, y) = G(x, y) - \frac{1}{2\pi} \log \frac{1}{|x - y|}. \quad (1.8)$$

We recall that the starting point for the statistical mechanics derivation of mean field equations for stationary flows with many vortices is given by Kirchhoff’s point vortex model whose dynamics is governed by the Hamiltonian \mathcal{H}_N . The mean field equations (1.1)–(1.4) are then derived by statistical mechanics arguments letting $N \rightarrow \infty$, along some ideas in [8]. It is therefore expected that if the residual vanishing property holds, then solutions to (1.1)–(1.4) should concentrate at critical points for \mathcal{H}_N . Theorem 2 will rigorously establish this fact for equation (1.1).

In order to state our main results more precisely, we introduce some notation. Let (λ_k, v_k) be a solution sequence to (1.1). We define the blowup sets:

$$\begin{aligned} \mathcal{S}_\pm &= \{x_0 \in \overline{\Omega} \mid \text{there exists } x_k \in \Omega \text{ such that } x_k \rightarrow x_0 \text{ and } v_k(x_k) \rightarrow \pm\infty\}, \\ \mathcal{S} &= \mathcal{S}_+ \cup \mathcal{S}_-. \end{aligned}$$

We point out that our definition of \mathcal{S} allows the case $\mathcal{S} \cap \partial\Omega \neq \emptyset$. We further define

$$\alpha_-^* = \min \text{supp } \mathcal{P}, \quad \alpha_+^* = \max \text{supp } \mathcal{P} \quad (1.9)$$

where $\text{supp } \mathcal{P} = \{\alpha \in [-1, 1] \mid \mathcal{P}(N) > 0 \text{ for any neighborhood } N \text{ of } \alpha\}$ denotes the support of \mathcal{P} . For every $x_0 \in \mathcal{S}$ we set

$$\beta_\pm(x_0) = \begin{cases} |\alpha_\pm^*|^{-1} & \text{if } x_0 \in \mathcal{S}_\pm \\ 0 & \text{if } x_0 \notin \mathcal{S}_\pm \end{cases}. \quad (1.10)$$

With this notation, we have the following.

Theorem 1. *Let (λ_k, v_k) be a solution sequence for (1.1). Then, passing to a subsequence, we have the following alternatives.*

(I) *Compactness: $\limsup_{k \rightarrow \infty} \|v_k\|_\infty < +\infty$, that is, $\mathcal{S} = \emptyset$.*

Then, there exists $v \in H_0^1(\Omega)$ such that $v_k \rightarrow v$ in $H_0^1(\Omega)$ and v is a solution of (1.1).

(II) *Concentration: $\limsup_{k \rightarrow \infty} \|v_k\|_\infty = \infty$, that is, $\mathcal{S} \neq \emptyset$.*

Then, the following properties hold:

(i) *[Exclusion of boundary blowup points]:*

$$\mathcal{S} \cap \partial\Omega = \emptyset. \quad (1.11)$$

(ii) *It holds that $v_k \rightarrow v_0$ in $C_{loc}^2(\Omega \setminus \mathcal{S})$, where $G = G(x, y)$ is the Green function defined by (1.7) and*

$$v_0(x) = \sum_{x'_0 \in \mathcal{S}} (m_+(x'_0) - m_-(x'_0)) G(x, x'_0) \quad (1.12)$$

with $m_\pm(x'_0) \geq 4\pi$ for every $x'_0 \in \mathcal{S}$.

(iii) *[Mass relation]*

$$(m_+(x_0) - m_-(x_0))^2 = 8\pi(\beta_+(x_0)m_+(x_0) + \beta_-(x_0)m_-(x_0)) \quad (1.13)$$

for every $x_0 \in \mathcal{S}$, where $\beta_\pm(x_0)$ is defined in (1.10).

The mass relation (1.13) was first noticed for (1.4) in [20] in the special case where \mathcal{P} is given by $\mathcal{P} = \tau\delta_{+1} + (1 - \tau)\delta_{-1}$ ($\tau \in (0, 1)$). It was then derived for (1.4) with a general probability \mathcal{P} in [19]. In [25] a mass relation was established for a general equation including (1.1) and (1.4) as special cases, see Proposition 1 below.

Our second results is concerned with the location of the blow-up points.

Theorem 2. *Let (λ_k, v_k) be a solution sequence for (1.1) and suppose the alternative (II) in Theorem 1 occurs. Then, for every $x_0 \in \mathcal{S}$, we have*

$$\nabla \left[H(x, x_0) + \sum_{x'_0 \in \mathcal{S} \setminus \{x_0\}} \frac{(m_+(x'_0) - m_-(x'_0))}{(m_+(x_0) - m_-(x_0))} G(x, x'_0) \right] \Big|_{x=x_0} = 0. \quad (1.14)$$

We note that at a blow-up point we necessarily have $m_+(x_0) - m_-(x_0) \neq 0$ in view of the mass relation Theorem 1-II-(iii), so that the function in (1.14) is well-defined. Setting $\mathcal{S} = \{x_1, \dots, x_m\}$, (1.14) is equivalent to stating that (x_1, \dots, x_m) is a critical point for the Hamiltonian \mathcal{H}_N defined in (1.6) with $N = m = \text{card } \mathcal{S}$ and $r_i = m_+(x_i) - m_-(x_i)$, $i = 1, \dots, m$.

We observe that the properties stated in Theorem 1 and Theorem 2 are essentially properties of the following “model case”

$$\begin{cases} -\Delta v = \frac{\lambda}{2} \frac{\alpha_+^* e^{\alpha_+^* v} + \alpha_-^* e^{\alpha_-^* v}}{\int_\Omega (e^{\alpha_+^* v} + e^{\alpha_-^* v})} & \text{in } \Omega \\ v = 0 & \text{on } \partial\Omega \end{cases} \quad (1.15)$$

corresponding to $\mathcal{P} = (\delta_{\alpha_+^*} + \delta_{\alpha_-^*})/2$ (recall that $\alpha_-^* \leq 0$). This fact is evident in the “nondegenerate case” $\mathcal{P}(\{\alpha_+^*\}) > 0$ and $\mathcal{P}(\{\alpha_-^*\}) > 0$; technical care is needed in order to show that it holds true in the “degenerate case” $\mathcal{P}(\{\alpha_+^*\}) = \mathcal{P}(\{\alpha_-^*\}) = 0$. We note that sign-changing nodal solutions for (1.15) blowing up at two distinct points of Ω were recently constructed in [22].

This paper is organized as follows. In Section 2, we provide several preliminary lemmas. In particular, we construct a convenient conformal mapping X_0 which will be useful to “straighten the boundary” $\partial\Omega$ near a point $x_0 \in \partial\Omega$. Such a conformal mapping will allow us to reduce the boundary blow-up analysis to the case of a half-ball. We derive accurate estimates for the Green’s function \mathcal{G} for the half-ball. In Section 3 we exclude the existence of blow-up points on $\partial\Omega$. To this end, we argue by contradiction and we assume that $x_0 \in \partial\Omega$ is a blow-up point. By extending the Brezis-Merle arguments to the boundary via reflection arguments, we prove that a minimal mass is necessary for a boundary blow-up, so that x_0 is isolated. Consequently, we use the conformal mapping X_0 to pull-back the problem to the half-ball. Exploiting the estimates for \mathcal{G} we estimate the blow-up sequence in a small ball near the blow-up point x_0 . Then, a Pohozaev identity yields the desired contradiction. Section 4 is devoted to establishing the residual vanishing property. Extending Brezis-Merle type arguments, we show that if $m_+(x_0) - m_-(x_0) > 4\pi\beta_+$ or $m_-(x_0) - m_+(x_0) > 4\pi\beta_-$, then $\iint_{[-1,1] \times \Omega} e^{\alpha v_k} \mathcal{P}(d\alpha) dx \rightarrow \infty$ and consequently residual vanishing holds. On the other hand, we check that if $x_0 \in \mathcal{S}_+ \cap \mathcal{S}_-$, then the mass relation (1.13) implies $m_+(x_0) - m_-(x_0) > 4\pi\beta_+$ or $m_-(x_0) - m_+(x_0) > 4\pi\beta_-$, thus concluding the proof of residual vanishing. In Section 5 we prove Theorem 2. To this end, we comply the complex analysis argument developed by [33]. Finally, in Section 6 we derive the corresponding results for the problem defined on a compact Riemannian surface without boundary.

Notation Henceforth, we omit the notation dx and $\mathcal{P}(d\alpha)$ when it is clear from the context, and we do not distinguish the sequences appearing below from their subsequences. For the sake of simplicity, in what follows we denote $I = [-1, 1]$, $I_+ = [0, 1]$, $I_- = [-1, 0]$.

2 Preliminaries

Throughout this section, we use complex notations by identifying $x = (x_1, x_2) \in \mathbf{R}^2$ with $z = x_1 + ix_2 \in \mathbf{C}$ in the usual way, where i denotes the imaginary unit.

Fix $x_0 \in \partial\Omega$ and take $R_0 > 0$ such that $B_{2R_0}(x_0) \cap \partial\Omega$ is connected. We may assume that $x_0 = 0$. To study the problem near $x_0 = 0$, we fix $z_0 \in B_{R_0} \cap \Omega$ and take a conformal mapping $X_0 : B_{R_0} \cap \bar{\Omega} \rightarrow \bar{\mathbf{R}}_+^2$, where

$$\mathbf{R}_+^2 = \{(X_1, X_2) \mid X_2 > 0\},$$

such that

$$\begin{cases} X_0(0) = 0, & X_0(z_0) \notin B_3^+, \\ X_0(B_{R_0} \cap \Omega) \subset \mathbf{R}_+^2, \\ X_0(B_{R_0} \cap \Omega) \supset B_2 \cap \mathbf{R}_+^2, \\ X_0(B_{R_0} \cap \partial\Omega) \subset \partial\mathbf{R}_+^2, & X_0(B_{R_0} \cap \partial\Omega) \supset (-3, 3) \times \{0\}, \end{cases} \quad (2.1)$$

and that

$$A_0 \in C^1(\bar{B}_1^+), \quad A_0 \geq \delta_0 \quad (2.2)$$

for some $\delta_0 > 0$, where

$$B_r^+ = B_r \cap \mathbf{R}_+^2, \quad A_0(X) = |g'(X)|^2, \quad g = X^{-1}, \quad X = X_0$$

for $r > 0$. This is possible if $0 < R_0 \ll 1$, namely, the following lemma holds.

Lemma 2.1. *If $0 < R_0 \ll 1$ then there exists a conformal mapping $X_0 : B_{R_0} \cap \bar{\Omega} \rightarrow \bar{\mathbf{R}}_+^2$ satisfying (2.1)-(2.2).*

Proof. By the Carathéodory theorem, there exists $w_1 : \overline{B_{R_0} \cap \Omega} \rightarrow \bar{B}_1$ such that $w_1(z_0) = 0$, $w_1(B_{R_0} \cap \Omega) = B_1$, $w_1(\partial(B_{R_0} \cap \Omega)) = \partial B_1$, it is holomorphic in $B_{R_0} \cap \Omega$ and is homeomorphic on $\overline{B_{R_0} \cap \Omega}$. We may assume that

$$w_1(B_{R_0} \cap \partial\Omega) \ni -1, \quad w_1(B_{R_0} \cap \partial\Omega) \not\ni 1$$

by taking a suitable rotation. Let

$$w_2(z) = -i \frac{z+1}{z-1}, \quad w_3(z) = z - w_2 \circ w_1(0).$$

Then we find that $X_0 = L_0(w_3 \circ w_2 \circ w_1)$ satisfies (2.1) for $L_0 \gg 1$.

Since w_1 is injective and $|dw_1/dz| > 0$ in $B_{R_0} \cap \Omega$, there exists a function $H = H(z)$, which is holomorphic in $B_{R_0} \cap \Omega$ and continuous on $\overline{B_{R_0} \cap \Omega}$, such that

$$w_1(z) = (z - z_0)H(z), \quad H(\zeta) \neq 0$$

for $z \in B_{R_0} \cap \Omega$ and $\zeta \in \overline{B_{R_0} \cap \Omega}$. Note that there exists $0 < R'_0 < R_0$ such that $z_0 \notin B_{2R'_0}$ and $\log H$ is defined as a single-valued function on $\overline{B_{R'_0} \cap \Omega}$. For such an R'_0 , real-valued function

$$U(x) = \log |z - z_0| + \log |H(z)|$$

is the solution of

$$-\Delta U = 0 \text{ in } B_{R'_0} \cap \Omega, \quad U < 0 \text{ on } \partial B_{R'_0} \cap \Omega, \quad U = 0 \text{ on } B_{R'_0} \cap \partial\Omega.$$

Since

$$U \in C_{loc}^2(\overline{B_{R'_0} \cap \Omega} \setminus (\partial B_{R'_0} \cap \partial\Omega)) \quad (2.3)$$

by the elliptic regularities, and since $|\nabla U| > 0$ on $B_{R'_0} \cap \partial\Omega$ by the Hopf lemma, it holds that

$$\left| \frac{dw_1}{dz} \right| = |w_1| |\nabla U| = |\nabla U| > 0 \quad \text{on } B_{R'_0} \cap \partial\Omega.$$

Noting the conformality of w_1 , we conclude that

$$\left| \frac{dw_1}{dz} \right| > 0 \quad \text{in } \overline{B_{R'_0} \cap \Omega} \setminus (\partial B_{R'_0} \cap \partial\Omega). \quad (2.4)$$

Consequently, (2.3)-(2.4) and $X_0 = L_0(w_3 \circ w_2 \circ w_1)$ imply that X_0 satisfies (2.2) and $X_0 \in C^1(B_{R'_0} \cap \partial\Omega)$ for $L_0 \gg 1$.

Finally, we retake R_0 by $R_0 = R'_0/2$ if needs, and obtain the desired conformal mapping X_0 for $L_0 \gg 1$. \square

Now, fix $0 < R_0 \ll 1$ and let (λ, v) be a solution of (1.1). For every function φ defined on $B_{R_0} \cap \bar{\Omega}$, we put

$$\hat{\varphi}(X) = \varphi \circ g(X), \quad g = X^{-1}, \quad X = X_0.$$

Then we have

$$\begin{cases} -\Delta_X \hat{v} = c A_0(X) \int_I \alpha e^{\alpha \hat{v}} \mathcal{P}(d\alpha) & \text{in } B_2^+ \\ \hat{v} = 0 & \text{on } B_2 \cap \partial \mathbf{R}_+^2, \end{cases}$$

where

$$c = \frac{\lambda}{\iint_{I \times \Omega} e^{\alpha' v}}.$$

In the proof of Theorem 1, we use the Brezis-Merle inequality ([4]).

Lemma 2.2. *Let $D \subset \mathbf{R}^2$ be a bounded domain, and let $u = u(x)$ be a solution of*

$$-\Delta u = f \text{ in } D, \quad u = 0 \text{ on } \partial D$$

with $f \in L^1(D)$. Then, for $\delta \in (0, 4\pi)$, we have

$$\int_D \exp \left(\frac{(4\pi - \delta)|u(x)|}{\|f\|_{L^1(D)}} \right) \leq \frac{4\pi^2}{\delta} (\text{diam}(D))^2,$$

where $\text{diam}(D) = \sup_{x, y \in D} |x - y|$.

Finally, we provide the estimates concerning the Green function $\mathcal{G} = \mathcal{G}(x, y)$ for B_1^+ defined by

$$\mathcal{G}(x, y) = -\frac{1}{2\pi} \log \left| \frac{(z - w)(1 - zw)}{(z - \bar{w})(1 - z\bar{w})} \right|, \quad (2.5)$$

that is,

$$-\Delta \mathcal{G}(\cdot, y) = \delta_y \quad \text{in } B_1^+, \quad \mathcal{G}(\cdot, y) = 0 \quad \text{on } \partial B_1^+,$$

where $z = x_1 + ix_2$ and $w = y_1 + iy_2$.

Lemma 2.3. *Given $0 < \delta \ll 1$, we have*

$$|\mathcal{G}(x, y)| \leq \frac{1}{2\pi} \log \frac{2(1 + \delta)}{1 - \delta} \quad \text{for } (x, y) \in (B_1^+ \setminus B_{3\delta}^+) \times B_\delta^+, \quad (2.6)$$

$$|\nabla_x \mathcal{G}(x, y)| \leq \frac{2}{\pi(1 - \delta)^2} \quad \text{for } (x, y) \in (B_1^+ \setminus B_{\sqrt{\delta} + \delta}) \times B_\delta^+. \quad (2.7)$$

Proof. In this proof, we again use $z = x_1 + ix_2$ and $w = y_1 + iy_2$ in complex notations. Note that $|\nabla_x \mathcal{G}(x, y)| = |\frac{d}{dz} \mathcal{H}(z, w)|$ and

$$\frac{d}{dz} \mathcal{H}(z, w) = -\frac{1 - |w|^2}{2\pi} \cdot \frac{(w - \bar{w})(1 - z^2)}{(z - w)(z - \bar{w})(1 - zw)(1 - z\bar{w})},$$

where

$$\mathcal{H}(z, w) = -\frac{1}{2\pi} \log \frac{(z - w)(1 - zw)}{(z - \bar{w})(1 - z\bar{w})}.$$

[Proof of (2.6)] For $(z, w) \in (B_1^+ \setminus B_{3\delta}^+) \times B_\delta^+$, we compute

$$\begin{aligned} \left| \frac{(z - w)(1 - zw)}{(z - \bar{w})(1 - z\bar{w})} \right| &\leq \frac{|z| + |w|}{|z| - |w|} \cdot \frac{1 + |w||z|}{1 - |w||z|} \leq \frac{|z| + \delta}{|z| - \delta} \cdot \frac{1 + \delta|z|}{1 - \delta|z|} \\ &\leq \frac{1 + \delta}{1 - \delta} \left(1 + \frac{2\delta}{|z| - \delta} \right) \leq \frac{2(1 + \delta)}{1 - \delta} \end{aligned}$$

Similarly, for $(z, w) \in (B_1^+ \setminus B_{3\delta}^+) \times B_\delta^+$, we compute

$$\left| \frac{(z - w)(1 - zw)}{(z - \bar{w})(1 - z\bar{w})} \right| \geq \left(\frac{2(1 + \delta)}{1 - \delta} \right)^{-1}.$$

[Proof of (2.7)] For $(z, w) \in (B_1^+ \setminus B_{\sqrt{\delta}+\delta}^+) \times B_\delta^+$, we compute

$$\begin{aligned} \left| \frac{d\mathcal{H}}{dz}(z, w) \right| &\leq \frac{1}{2\pi} \cdot \frac{4|w|}{(|z| - |w|)^2(1 - |z||w|)^2} \\ &\leq \frac{2\delta}{\pi(|z| - \delta)^2(1 - \delta|z|)^2} \leq \frac{2}{\pi(1 - \delta)^2}. \end{aligned}$$

Hence, the desired estimates are shown. \square

3 Proof of Theorem 1: proof of (1.11)

To begin with, we collect in Proposition 1 below some results from [25] which are used in the proof of Theorem 1. Although such results were actually derived in the case of a compact manifold, the extension to the case of *interior* blow-up points for the Dirichlet problem is straightforward since the arguments are local in nature.

We introduce the measure functions $\nu_{k,\pm} = \nu_{k,\pm}(dx) \in \mathcal{M}(\bar{\Omega})$ and $\mu_k = \mu_k(d\alpha dx) \in \mathcal{M}(I \times \bar{\Omega})$ defined by

$$\nu_{k,\pm} = \lambda_k \int_{I_\pm} |\alpha| V(\alpha, v_k) e^{\alpha v_k} \mathcal{P}(d\alpha), \quad (3.1)$$

$$\mu_k(d\alpha dx) = \lambda_k V(\alpha, v_k) e^{\alpha v_k} \mathcal{P}(d\alpha) dx, \quad (3.2)$$

respectively, where $V(\alpha, v) = \left(\iint_{[-1,1] \times \Omega} e^{\alpha v} \mathcal{P}(d\alpha) dx \right)^{-1}$ if v_k satisfies (1.1) and $V(\alpha, v) = \left(\int_\Omega e^{\alpha v} dx \right)^{-1}$ if v_k satisfies (1.4). We recall that $I_+ = [0, 1]$, $I_- = [-1, 0]$. Then, Theorem 2.1 and Theorem 2.2 in [25] are readily adapted to the domain case to yield the following basic blow-up properties.

Proposition 1 ([25], common blow-up properties for (1.1) and (1.4)). *Let (λ_k, v_k) be a solution sequence of (1.1) or (1.4) with $\lambda_k \rightarrow \lambda_0$ for some $\lambda_0 \geq 0$. Assume that*

$$\mathcal{S} \cap \partial\Omega = \emptyset$$

holds true. Then, passing to a subsequence, we have the following alternatives.

(I) *Compactness: $\limsup_{k \rightarrow \infty} \|v_k\|_\infty < +\infty$, that is, $\mathcal{S} = \emptyset$.*

Then, there exists $v \in H_0^1(\Omega)$ such that $v_k \rightarrow v$ in $H_0^1(\Omega)$ and v is a solution of (1.1) or (1.4).

(II) *Concentration: $\limsup_{k \rightarrow \infty} \|v_k\|_\infty = \infty$, that is, $\mathcal{S} \neq \emptyset$.*

Then, \mathcal{S} is finite and the following properties a)-c) hold:

a) *There exists $0 \leq s_\pm \in L^1(\Omega) \cap L_{loc}^\infty(\Omega \setminus \mathcal{S})$ such that*

$$\nu_{k,\pm} \xrightarrow{*} \nu_\pm = s_\pm + \sum_{x_0 \in \mathcal{S}_\pm} m_\pm(x_0) \delta_{x_0} \quad \text{in } \mathcal{M}(\bar{\Omega}), \quad (3.3)$$

with $m_\pm(x_0) \geq 4\pi$ for every $x_0 \in \mathcal{S}_\pm$, where $\delta_x \in \mathcal{M}(\Omega)$, $x \in \Omega$, denotes the Dirac measure centered at x .

b) *There exist $\zeta_{x_0} \in \mathcal{M}(I)$ and $0 \leq r \in L^1(I \times \Omega)$ such that*

$$\mu_k \xrightarrow{*} \mu = \mu(d\alpha dx) = r(\alpha, x) \mathcal{P}(d\alpha) dx + \sum_{x_0 \in \mathcal{S}} \zeta_{x_0}(d\alpha) \delta_{x_0}(dx) \quad \text{in } \mathcal{M}(I \times \bar{\Omega}). \quad (3.4)$$

c) *For every $x_0 \in \mathcal{S}$, we have*

$$8\pi \int_I \zeta_{x_0}(d\alpha) = \left(\int_I \alpha \zeta_{x_0}(d\alpha) \right)^2 \quad (3.5)$$

$$m_\pm(x_0) = \int_{I_\pm} |\alpha| \zeta_{x_0}(d\alpha), \quad s_\pm(x) = \int_{I_\pm} |\alpha| r(\alpha, x) \mathcal{P}(d\alpha), \quad (3.6)$$

where $m_\pm(x_0)$ is as in (3.3). Moreover, for every $x_0 \in \mathcal{S}_\pm \setminus \mathcal{S}_\mp$, it holds that

$$m_\mp(x_0) = \int_{I_\mp} |\alpha| \zeta_{x_0}(d\alpha) = 0. \quad (3.7)$$

To prove Theorem 1, we see from Proposition 1 that it suffices to show (1.11), (1.12) and (1.13) under the assumption that $\mathcal{S} \neq \emptyset$.

In the remainder of this section, we shall prove (1.11) by contradiction under the assumption that $\mathcal{S} \cap \partial\Omega \neq \emptyset$.

Let (λ_k, v_k) be a solution sequence of (1.1). The starting point for the proof of (1.11) is the following lemma based on the Brezis-Merle inequality, see Lemma 2.2.

Lemma 3.1. *For any $x_0 \in \mathcal{S}_\pm \cap \partial\Omega$, it holds that*

$$\lim_{r \downarrow 0} \limsup_{k \rightarrow \infty} \nu_{k,\pm}(B_r(x_0) \cap \Omega) \geq 4\pi, \quad (3.8)$$

where $\nu_{k,\pm}$ is as in (3.1).

Proof. Fix $x_0 \in \mathcal{S} \cap \partial\Omega$. We may assume that $x_0 = 0$. Assume that (3.8) is false. Then, there exist $0 < \varepsilon_0, r_0 \ll 1$ such that $B_{3r_0} \cap (\mathcal{S} \cap \Omega) = \emptyset$ and

$$\nu_{k,\pm}(B_{3r_0} \cap \Omega) \leq 4\pi - 2\varepsilon_0 \quad (3.9)$$

for $k \gg 1$. We decompose v_k as $v_k = v_k^+ - v_k^-$, where v_k^\pm is the solution of

$$-\Delta v_k^\pm = \nu_{k,\pm} \text{ in } \Omega, \quad v_k^\pm = 0 \text{ on } \partial\Omega.$$

Let $v_{k,1}^\pm$ and $v_{k,2}^\pm$ be the solutions of

$$\begin{aligned} -\Delta v_{k,1}^\pm &= \nu_{k,\pm} \text{ in } B_{2r_0} \cap \Omega, \quad v_{k,1}^\pm = 0 \text{ on } \partial(B_{2r_0} \cap \Omega), \\ -\Delta v_{k,2}^\pm &= 0 \text{ in } B_{2r_0} \cap \Omega, \quad v_{k,2}^\pm = v_k^\pm \text{ on } \partial(B_{2r_0} \cap \Omega), \end{aligned}$$

respectively. Then it holds that $v_k^\pm = v_{k,1}^\pm + v_{k,2}^\pm$ on $\bar{B}_{2r_0} \cap \bar{\Omega}$. In addition, by the maximum principle and the L^1 -estimates (see [6]), we have

$$v_k^\pm \geq 0 \text{ on } \bar{\Omega}, \quad v_{k,1}^\pm \geq 0 \text{ on } \bar{B}_{2r_0} \cap \bar{\Omega}, \quad (3.10)$$

$$\|v_k^\pm\|_{L^1(\Omega)} + \|v_{k,1}^\pm\|_{L^1(B_{2r_0} \cap \Omega)} \leq C_1 \quad (3.11)$$

for some $C_1 > 0$ independent of k and $0 < r_0 \ll 1$.

To estimate $v_{k,2}^\pm = v_{k,2}^\pm(x)$, we take a conformal mapping $X : B_{2r_0} \cap \bar{\Omega} \rightarrow \bar{\mathbf{R}}^2$ as in Section 2 for $R_0 = 2r_0$. Let $\tilde{v}_{k,2}^\pm$ be the odd extension of $v_{k,2}^\pm \circ X^{-1}$. Then it holds that

$$-\Delta_X \tilde{v}_{k,2}^\pm = 0 \text{ in } B_1, \quad \tilde{v}_{k,2}^\pm = 0 \text{ on } B_1 \cap \partial\mathbf{R}_+^2,$$

and the mean value theorem and (3.11) admit $C_2 > 0$, independent of k , such that

$$\|v_{k,2}^\pm\|_{L^\infty(B_{r_1} \cap \Omega)} \leq C_2 \quad (3.12)$$

for $k \gg 1$ and for some $0 < r_1 < 2r_0$.

On the other hand, Lemma 2.2 and (3.9) yield

$$\int_{B_{2r_0} \cap \Omega} \exp\left(\frac{4\pi - \varepsilon_0}{4\pi - 2\varepsilon_0} v_{k,1}^\pm\right) \leq \frac{4\pi^2}{\varepsilon_0} (4r_0)^2 \quad (3.13)$$

for $k \gg 1$. Combining (3.10) and (3.12)-(3.13), and noting that $\pm v_k \leq v_k^\pm$, we obtain

$$\begin{aligned} \int_{B_{r_1} \cap \Omega} e^{p_0 \alpha v_k} &\leq \int_{B_{r_1} \cap \Omega} e^{p_0 v_k^\pm} \\ &\leq \int_{B_{r_1} \cap \Omega} e^{p_0 (v_{k,1}^\pm + |v_{k,2}^\pm|)} \leq \frac{(8\pi r_0)^2}{\varepsilon_0} \cdot e^{p_0 C_2} \end{aligned}$$

for any $\alpha \in I_\pm$, where $p_0 = (4\pi - \varepsilon_0)/(4\pi - 2\varepsilon_0) > 1$. This estimate means

$$\|\nu_{k,\pm}\|_{L^{p_0}(B_{r_1} \cap \Omega)}^{p_0} \leq C_3$$

for any $k \gg 1$ and for some $C_3 > 0$ independent of k . Consequently, the boundary L^p -estimate guarantees the uniform boundedness of v_k^\pm in $B_{r_1/2} \cap \Omega$, which

contradicts $0 \in \mathcal{S}_\pm$. \square

Now we fix $x_0 \in \mathcal{S} \cap \partial\Omega$. We may assume that $x_0 = 0$. By virtue of Lemma 3.1, we see that \mathcal{S} is finite, and hence there exists $0 < R_0 \ll 1$ such that $B_{2R_0} \cap \mathcal{S} = \{0\}$. After passing to a subsequence, we set

$$m(0) = \lim_{r \downarrow 0} \lim_{k \rightarrow \infty} \nu_{k,+}(B_r \cap \Omega) + \nu_{k,-}(B_r \cap \Omega) \geq 4\pi.$$

Given $0 < \varepsilon \ll 1$, there exists $r_\varepsilon \in (0, 2R_0)$ such that

$$\lim_{k \rightarrow \infty} \nu_{k,+}(B_{r_\varepsilon} \cap \Omega) + \nu_{k,-}(B_{r_\varepsilon} \cap \Omega) \leq m(0) + \varepsilon/4.$$

We transform the problem into the one on B_1^+ by taking the conformal mapping $X : B_{r_\varepsilon} \cap \bar{\Omega} \rightarrow \bar{\mathbf{R}}_+^2$ as in Section 2 for $R_0 = r_\varepsilon$. For simplicity, we shall denote \hat{v}_k , X , ∇_X and Δ_X by v_k , x , ∇ and Δ , respectively. Under these agreements, we obtain

$$\begin{cases} -\Delta v_k = \nu_k & \text{in } B_2^+ \\ v_k = 0 & \text{on } B_2 \cap \partial\mathbf{R}_+^2, \end{cases} \quad (3.14)$$

where

$$\nu_k = \nu_{k,+} - \nu_{k,-} = c_k A_0(x) \int_I \alpha e^{\alpha v_k} \mathcal{P}(d\alpha), \quad c_k = \frac{\lambda_k}{\iint_{I \times \Omega} e^{\alpha' v_k}}.$$

Note that

$$m(0) = \lim_{r \downarrow 0} \lim_{k \rightarrow \infty} \nu_{k,+}(B_r^+) + \nu_{k,-}(B_r^+) \geq 4\pi, \quad (3.15)$$

$$\lim_{k \rightarrow \infty} \nu_{k,+}(B_1^+) + \nu_{k,-}(B_1^+) \leq m(0) + \varepsilon/4, \quad (3.16)$$

and that there exists $r_k \downarrow 0$ such that

$$\lim_{k \rightarrow \infty} \nu_{k,+}(B_{r_k}^+) + \nu_{k,-}(B_{r_k}^+) = m(0). \quad (3.17)$$

We now show the crucial estimate.

Lemma 3.2. *There exists $C_4 > 0$, independent of δ , such that*

$$\limsup_{k \rightarrow \infty} \|v_k\|_{W^{1,\infty}(B_1^+ \setminus B_\delta^+)} \leq C_4$$

for any $0 < \delta \ll 1$.

Proof. The proof is split into five steps.

Step 1. Let w_k and h_k be the solutions of

$$\begin{cases} -\Delta w_k = \nu_k & \text{in } B_1^+ \\ w_k = 0 & \text{on } \partial B_1^+, \end{cases} \quad \begin{cases} -\Delta h_k = 0 & \text{in } B_1^+ \\ h_k = v_k & \text{on } \partial B_1^+, \end{cases}$$

respectively, so that $v_k = w_k + h_k$. Then, there exists $C_5 > 0$, independent of k , such that

$$\|h_k\|_{W^{1,\infty}(B_1^+)} \leq C_5. \quad (3.18)$$

In fact, the odd extension $\tilde{h}_k = \tilde{h}_k(x)$ of h_k is the solution of

$$-\Delta \tilde{h}_k = 0 \text{ in } B_1, \quad \tilde{h}_k = \tilde{v}_k \text{ on } \partial B_1,$$

where $\tilde{v}_k = \tilde{v}_k(x)$ is the odd one of v_k . Therefore, $\|\tilde{h}_k\|_{W^{1,\infty}(B_1)}$ is uniformly bounded by the maximum principle and the uniform boundedness of $\|v_k\|_{W^{1,\infty}(\partial B_1^+)}$, which means (3.18).

Step 2. Let $w_{k,1}$ and $w_{k,2}$ be the solutions of

$$\begin{cases} -\Delta w_{k,1} = \chi_{B_{r_k}^+} \nu_k & \text{in } B_1^+ \\ w_{k,1} = 0 & \text{on } \partial B_1^+, \end{cases} \quad \begin{cases} -\Delta w_{k,2} = \chi_{B_1^+ \setminus B_{r_k}^+} \nu_k & \text{in } B_1^+ \\ w_{k,2} = 0 & \text{on } \partial B_1^+, \end{cases}$$

so that $w_k = w_{k,1} + w_{k,2}$, recall that r_k satisfies (3.17), where χ_A denotes the characteristic function of $A \subset \mathbf{R}^2$. Let $\mathcal{G} = \mathcal{G}(x, y)$ be the Green function for B_1^+ defined by (2.5). Then, the representation formula

$$w_{k,1}(x) = \int_{B_1^+} \mathcal{G}(x, y) \chi_{B_{r_k}^+}(y) \nu_k(y) dy = \int_{B_{r_k}^+} \mathcal{G}(x, y) \nu_k(y) dy$$

shows

$$\lim_{k \rightarrow \infty} w_{k,1}(x) = 0 \quad (3.19)$$

for every $x \in \bar{B}_1^+$, since $r_k \downarrow 0$, $\lim_{y \rightarrow \partial B_1^+} \mathcal{G}(x, y) = 0$ and $\mathcal{G}(x, \cdot) \in C(\bar{\Omega} \setminus \{x\})$ for $x \in B_1^+$, and $\|\nu_k\|_{L^1(B_1^+)}$ is uniformly bounded.

Fix $0 < \varepsilon \ll 1$. Then, the representation formula of $w_{k,1}$ above and (2.6) admit $C_6 > 0$, independent of ε , such that

$$\limsup_{k \rightarrow \infty} \|w_{k,1}\|_{L^\infty(B_1^+ \setminus B_{\varepsilon/2}^+)} \leq C_6. \quad (3.20)$$

Here, we consider the problem

$$\begin{cases} -\Delta \psi_k = 0 & \text{in } B_1^+ \setminus B_\varepsilon^+ \\ \psi_k = w_{k,1} & \text{on } \partial B_\varepsilon \cap \mathbf{R}_+^2 \\ \psi_k = 0 & \text{on } \partial(B_1^+ \setminus B_\varepsilon^+) \setminus (\partial B_\varepsilon \cap \mathbf{R}_+^2). \end{cases}$$

Note that $\psi_k = w_{k,1}$ on $\overline{B_1^+ \setminus B_\varepsilon^+}$ for $k \gg 1$ since $r_k \downarrow 0$. Let $\tilde{\psi}_k = \tilde{\psi}_k(x)$ and $\tilde{w}_{k,1} = \tilde{w}_{k,1}(x)$ be the odd extensions of ψ_k and $w_{k,1}$, respectively. Then, we have

$$\begin{cases} -\Delta \tilde{\psi}_k = 0 & \text{in } B_1 \setminus B_\varepsilon \\ \tilde{\psi}_k = \tilde{w}_{k,1} & \text{on } \partial B_\varepsilon \\ \tilde{\psi}_k = 0 & \text{on } \partial B_1 \end{cases}$$

and use the maximum principle and (3.20) to find that there exists $C_7 > 0$, independent of ε , such that

$$\limsup_{k \rightarrow \infty} \|\tilde{w}_{k,1}\|_{L^\infty(B_1 \setminus B_\varepsilon)} = \limsup_{k \rightarrow \infty} \|\tilde{\psi}_k\|_{L^\infty(B_1 \setminus B_\varepsilon)} \leq C_7,$$

where we have used the property that $\psi_k = w_{k,1}$ on $\overline{B_1^+ \setminus B_\varepsilon^+}$ for $k \gg 1$. Thus, the elliptic regularity yields $C_{8,\varepsilon} > 0$ such that

$$\limsup_{k \rightarrow \infty} \|\tilde{w}_{k,1}\|_{W^{1,\infty}(B_1 \setminus B_{2\varepsilon})} = \limsup_{k \rightarrow \infty} \|\tilde{\psi}_k\|_{W^{1,\infty}(B_1 \setminus B_{2\varepsilon})} \leq C_{8,\varepsilon}.$$

Hence, the Arzelà-Ascoli theorem and (3.19) guarantee that

$$w_{k,1} \rightarrow 0 \quad \text{in } C(\bar{B}_1^+ \setminus B_{2\varepsilon}).$$

Since ε is arbitrary, we conclude that

$$w_{k,1} \rightarrow 0 \quad \text{in } C_{loc}(\bar{B}_1^+ \setminus \{0\}). \quad (3.21)$$

Step 3. Now (3.15)-(3.16) and Lemma 2.2 show

$$\int_{B_1^+} e^{2|w_{k,2}|} = \int_{B_1^+} e^{\frac{4\pi - (4\pi - \varepsilon)}{\varepsilon/2} |w_{k,2}|} \leq \int_{B_1^+} e^{\frac{4\pi - (4\pi - \varepsilon)}{\|\chi_{B_1^+ \setminus B_{r_k}^+} \nu_k\|_{L^1(B_1^+)}} |w_{k,2}|} \leq 8\pi \quad (3.22)$$

for $k \gg 1$. Summarizing (3.18), (3.21)-(3.22) and the uniform boundedness of $c_k A_0$, we obtain $C_9 > 0$, independent of δ , such that

$$\limsup_{k \rightarrow \infty} \|\nu_k\|_{L^2(B_1^+ \setminus B_\delta^+)} \leq C_9 \quad (3.23)$$

for any $0 < \delta \ll 1$.

Step 4. In this step, we shall derive the L^∞ -estimates of v_k . Given $0 < \delta \ll 1$, let $z_{k,1}^\delta$ and $z_{k,2}^\delta$ be the solutions of

$$\begin{cases} -\Delta z_{k,1}^\delta = \chi_{B_\delta^+} \nu_k & \text{in } B_1^+ \\ z_{k,1}^\delta = 0 & \text{on } \partial B_1^+, \end{cases} \quad \begin{cases} -\Delta z_{k,2}^\delta = \chi_{B_1^+ \setminus B_\delta^+} \nu_k & \text{in } B_1^+ \\ z_{k,2}^\delta = 0 & \text{on } \partial B_1^+, \end{cases} \quad (3.24)$$

so that $w_k = z_{k,1}^\delta + z_{k,2}^\delta$. It follows from (3.23) and the elliptic regularity that there exists $C_{10} > 0$, independent of δ , such that

$$\limsup_{k \rightarrow \infty} \|z_{k,2}^\delta\|_{L^\infty(B_1^+)} \leq C_{10}. \quad (3.25)$$

Furthermore, the representation formula

$$z_{k,1}^\delta(x) = \int_{B_1^+} \mathcal{G}(x, y) \chi_{B_\delta^+}(y) \nu_k(y) dy = \int_{B_\delta^+} \mathcal{G}(x, y) \nu_k(y) dy$$

and (2.6) admit $C_{11} > 0$, independent of δ , such that

$$\limsup_{k \rightarrow \infty} \|z_{k,1}^\delta\|_{L^\infty(B_1^+ \setminus B_{3\delta}^+)} \leq C_{11}. \quad (3.26)$$

Since $0 < \delta \ll 1$ is arbitrary and since the constants in (3.18) and (3.25)-(3.26) are independent of δ , we conclude that there exists $C_{12} > 0$, independent of δ' , such that

$$\limsup_{k \rightarrow \infty} \|v_k\|_{L^\infty(B_1^+ \setminus B_{\delta'}^+)} \leq C_{12} \quad (3.27)$$

for any $0 < \delta' \ll 1$.

Step 5. In the final step, we shall derive the gradient estimates of v_k . Given $0 < \delta \ll 1$, we again use the decomposition $w_k = z_{k,1}^\delta + z_{k,2}^\delta$ or (3.24). From (3.27) and the uniform boundedness of $c_k A_0$, we see that there exists $C_{13} > 0$, independent of δ , such that

$$\limsup_{k \rightarrow \infty} \|\chi_{B_1^+ \setminus B_\delta^+} \nu_k\|_{L^\infty(B_1^+)} \leq C_{13}.$$

Thus, the elliptic regularity yields $C_{14} > 0$, independent of δ , such that

$$\limsup_{k \rightarrow \infty} \|z_{k,2}^\delta\|_{W^{1,\infty}(B_1^+)} \leq C_{14}. \quad (3.28)$$

By (2.7), (3.16) and the representation formula

$$\nabla z_{k,1}^\delta(x) = \int_{B_1^+} \chi_{B_\delta^+}(y) \nu_k(y) \nabla_x \mathcal{G}(x, y) dy = \int_{B_\delta^+} \nu_k(y) \nabla_x \mathcal{G}(x, y) dy$$

it holds that

$$|\nabla z_{k,1}^\delta(x)| \leq \frac{2(m(0) + \varepsilon)}{\pi(1 - \delta)^2} \quad (3.29)$$

for any $x \in B_1^+ \setminus B_{\sqrt{\delta}+\delta}^+$ and $k \gg 1$.

Since δ is arbitrary, we combine (3.28)-(3.29) and (3.18) to obtain $C_{15} > 0$, independent of δ' , such that

$$\limsup_{k \rightarrow \infty} \|\nabla v_k\|_{L^\infty(B_1^+ \setminus B_{\delta'}^+)} \leq C_{15}$$

for any $0 < \delta' \ll 1$. Hence, the desired gradient estimates are established. \square

We are now in a position to prove (1.11).

Proof of (1.11) At first, we note the following Pohozaev type identity, that is,

$$\begin{aligned} & r \int_{\partial B_r^+ \cap \mathbf{R}^2} \left(\frac{|\nabla v_k|^2}{2} - (\mathbf{n} \cdot \nabla v_k)^2 \right) - c_k A_0 e^{\alpha v_k} d\sigma \\ &= - \int_{B_r^+} \left\{ 2 + \left(x \cdot \frac{\nabla A_0}{A_0} \right) \right\} c_k A_0 \left(\int_I e^{\alpha v_k} \mathcal{P}(d\alpha) \right) dx \end{aligned} \quad (3.30)$$

for any $0 < r < 1$, where \cdot , \mathbf{n} and $d\sigma$ denote the usual inner product in \mathbf{R}^2 , the outward unit normal vector to the boundary, and the line element on the boundary, respectively. Identity (3.30) is shown by multiplying (3.14) by $x \cdot \nabla v_k$ without difficulty.

Next, organizing (3.30), the uniform boundedness of $c_k A_0$, (2.2) and Lemma 3.2, we find

$$O(r^2) = -(2 + O(r)) \lim_{k \rightarrow \infty} \int_{B_r^+} c_k A_0 \left(\int_I e^{\alpha v_k} \mathcal{P}(d\alpha) \right) dx \quad (3.31)$$

as $r \downarrow 0$ after taking $k \rightarrow \infty$ and passing to a subsequence. However, the right-hand side of (3.31) does not converge to 0 as $r \downarrow 0$ because of (3.15), which is a contradiction. The proof is complete. \square

4 Proof of Theorem 1: proof of (1.12) and (1.13)

In this section, we shall give the proof of (1.12) and (1.13) under the assumption that the alternative **(II)** in Proposition 1 occurs. Note that $\mathcal{S} \cap \partial\Omega = \emptyset$ as shown in the previous section. Therefore, (3.3)-(3.7) now hold.

At first we shall prove (1.13). Suppose that the following propositions hold.

$$x_0 \in \mathcal{S}_\pm \setminus \mathcal{S}_\mp \Rightarrow \text{supp } \zeta_{x_0} = \{\alpha_\pm^*\}, \quad (4.1)$$

$$x_0 \in \mathcal{S}_+ \cap \mathcal{S}_- \Rightarrow \text{supp } \zeta_{x_0} = \{\alpha_+^*, \alpha_-^*\}, \quad (4.2)$$

where α_\pm^* is as in (1.9). Note that $\alpha_\pm^* \neq 0$ if $\mathcal{S}_\pm \neq \emptyset$. Then, (3.5)-(3.7) imply that

$$m_\pm(x_0) = 8\pi\beta_\pm, \quad m_\mp(x_0) = 0$$

if $x_0 \in \mathcal{S}_\pm \setminus \mathcal{S}_\mp$, and that

$$(m_+(x_0) - m_-(x_0))^2 = 8\pi(\beta_+m_+(x_0) + \beta_-m_-(x_0))$$

if $x_0 \in \mathcal{S}_+ \cap \mathcal{S}_-$, where β_\pm is as in (1.10). Therefore (4.1)-(4.2) assure (1.13), and hence the proof of (1.13) is reduced to showing (4.1)-(4.2).

Proof of (1.13) It suffices to prove (4.1)-(4.2) as we have seen above. We shall only give the proof of (4.2) here since that of (4.1) is similar.

We now fix $x_0 \in \mathcal{S}_+ \cap \mathcal{S}_-$. Then it holds that $\alpha_+^* > 0 > \alpha_-^*$. Furthermore, we find that the proof of (4.2) is reduced to proving that for any $0 < \varepsilon \ll 1$, there exists $C_1 > 0$ such that

$$\left\| \frac{e^{\alpha v_k}}{\iint_{I \times \Omega} e^{\alpha' v_k}} \right\|_{L^{p_\varepsilon}(\Omega)} \leq C_1 \quad (4.3)$$

for any k and $\alpha \in [\alpha_-^* + 2\varepsilon, \alpha_+^* - 2\varepsilon]$, where

$$p_\varepsilon = \min \left\{ \frac{\alpha_+^* - \varepsilon}{\alpha_+^* - 2\varepsilon}, \frac{\alpha_-^* + \varepsilon}{\alpha_-^* + 2\varepsilon} \right\} > 1.$$

In fact, if (4.3) holds then $\zeta_{x_0}([\alpha_-^* + 2\varepsilon, \alpha_+^* - 2\varepsilon]) = 0$ for any $0 < \varepsilon \ll 1$, which implies (4.2). Inequality (4.3) is obvious for $\alpha = 0$.

Given $\alpha \in (0, \alpha_+^* - 2\varepsilon]$, we have

$$\int_\Omega e^{p_\varepsilon \alpha v_k} \leq |\Omega|^{1 - \frac{\alpha p_\varepsilon}{\beta}} \left(\int_\Omega e^{\beta v_k} \right)^{\frac{\alpha p_\varepsilon}{\beta}} \leq |\Omega| + \int_\Omega e^{\beta v_k} \quad (4.4)$$

for any $\beta \in [\alpha_+^* - \varepsilon, \alpha_+^*]$. Note that $\beta/(\alpha p_\varepsilon) \geq 1$ for any $\beta \in [\alpha_+^* - \varepsilon, \alpha_+^*]$. Then, we combine (4.4) with $\mathcal{P}([\alpha_+^* - \varepsilon, \alpha_+^*]) > 0$ to obtain

$$\int_\Omega e^{p_\varepsilon \alpha v_k} \leq |\Omega| + \frac{1}{\mathcal{P}([\alpha_+^* - \varepsilon, \alpha_+^*])} \iint_{I \times \Omega} e^{\alpha' v_k} \mathcal{P}(d\alpha') dx,$$

that is,

$$\left\| \frac{e^{\alpha v_k}}{\iint_{I \times \Omega} e^{\alpha' v_k}} \right\|_{L^{p_\varepsilon}(\Omega)}^{p_\varepsilon} \leq \frac{|\Omega|}{\left(\iint_{I \times \Omega} e^{\alpha' v_k} \right)^{p_\varepsilon}} + \frac{1}{\mathcal{P}([\alpha_+^* - \varepsilon, \alpha_+^*]) \left(\iint_{I \times \Omega} e^{\alpha' v_k} \right)^{p_\varepsilon - 1}} \quad (4.5)$$

for any $\alpha \in (0, \alpha_+^* - 2\varepsilon]$. The similar argument yields

$$\left\| \frac{e^{\alpha v_k}}{\iint_{I \times \Omega} e^{\alpha' v_k}} \right\|_{L^{p_\varepsilon}(\Omega)}^{p_\varepsilon} \leq \frac{|\Omega|}{\left(\iint_{I \times \Omega} e^{\alpha' v_k} \right)^{p_\varepsilon}} + \frac{1}{\mathcal{P}([\alpha_-^*, \alpha_-^* + \varepsilon]) \left(\iint_{I \times \Omega} e^{\alpha' v_k} \right)^{p_\varepsilon - 1}} \quad (4.6)$$

for any $\alpha \in [\alpha_-^* + 2\varepsilon, 0)$.

On the other hand, for $\omega \subset \subset \Omega \setminus \mathcal{S}$, there exists $C_{2,\omega} > 0$ such that

$$\|v_k\|_{L^\infty(\omega)} \leq C_{2,\omega}$$

for any k , and thus

$$\iint_{I \times \Omega} e^{\alpha' v_k} \geq \iint_{I \times \omega} e^{\alpha' v_k} \geq |\omega| e^{-C_{2,\omega}} > 0 \quad (4.7)$$

for any k . Consequently, (4.3) follows from (4.5)-(4.7) \square

It is left to prove (1.12). For the purpose, we prepare the following lemma.

Lemma 4.1. *For $x_0 \in \mathcal{S}$, if*

$$m_+(x_0) - m_-(x_0) > 4\pi\beta_+ \quad \text{or} \quad m_-(x_0) - m_+(x_0) > 4\pi\beta_-, \quad (4.8)$$

then

$$\lim_{k \rightarrow \infty} \iint_{I \times \Omega} e^{\alpha v_k} \mathcal{P}(d\alpha) dx = +\infty. \quad (4.9)$$

Proof. We shall prove the lemma only for the case that $x_0 \in \mathcal{S}_+ \cap \mathcal{S}_-$ and $m_+(x_0) - m_-(x_0) > 4\pi\beta_+$, since the lemma for the other cases are similarly shown. In the following, the proof is divided into four steps.

Step 1. Fix $x_0 \in \mathcal{S}_+ \cap \mathcal{S}_-$, and assume that (4.9) is false to prove the lemma by contradiction. Since $\mathcal{S} \subset \Omega$ now, there exists $0 < r_0 \ll 1$ such that $B_{2r_0} \Subset \Omega$ and $B_{2r_0} \cap \mathcal{S} = \{x_0\}$. We may assume that $x_0 = 0$. In the following, we consider the problem in B_{2r_0} , so that

$$-\Delta v_k = \lambda_k \int_I \frac{\alpha e^{\alpha v_k}}{\iint_{I \times \Omega} e^{\alpha' v_k} \mathcal{P}(d\alpha') dx} \mathcal{P}(d\alpha) \quad \text{in } B_{2r_0}.$$

By retaking $0 < r_0 \ll 1$, we can take $0 < \varepsilon \ll 1$ such that

$$\frac{(\alpha_+^* - 2\varepsilon)(\alpha_+^* - \varepsilon)}{\varepsilon} \leq \frac{2\pi}{\|s_-\|_{L^1(B_{r_0})}}, \quad \alpha_+^* > 2\varepsilon, \quad (4.10)$$

$$(\alpha_+^* - 2\varepsilon)(m_+(x_0) - m_-(x_0)) > 4\pi, \quad (4.11)$$

since $s_- \in L^1(\Omega)$ and $m_+(0) - m_-(0) > 4\pi\beta_+$, recall (1.10).

Carefully reading [25] shows that there exists $v \in H_{loc}^1(\Omega \setminus \mathcal{S})$ such that

$$v_k \rightarrow v \quad \text{in } H_{loc}^1(\Omega \setminus \mathcal{S}) \quad (4.12)$$

and

$$\begin{cases} -\Delta v = (s_+ - s_-) + \sum_{y_0 \in \mathcal{S}_+} m_+(y_0) \delta_{y_0} - \sum_{y_0 \in \mathcal{S}_-} m_-(y_0) \delta_{y_0} & \text{in } \Omega \\ v = 0 & \text{on } \partial\Omega. \end{cases} \quad (4.13)$$

Step 2. Let $z = z(x)$ be the very weak solution of

$$\begin{cases} -\Delta z = -s_- + (m_+(x_0) - m_-(x_0))\delta_0 & \text{in } B_{r_0} \\ z = b_0 := \min_{\partial B_{r_0}} v & \text{on } \partial B_{r_0}. \end{cases},$$

see [32] for the concept of very weak solutions. Since $v = v(x)$ satisfies

$$-\Delta v = (s_+ - s_-) + (m_+(x_0) - m_-(x_0))\delta_0 \quad \text{in } B_{r_0},$$

the maximum principle and $s_+ \geq 0$ imply

$$z \leq v \quad \text{a.e. in } B_{r_0}. \quad (4.14)$$

Furthermore, we decompose $z = z(x)$ as $z = z_1 + z_2$, where z_1 and z_2 are the solutions of

$$\begin{aligned} -\Delta z_1 &= (m_+(x_0) - m_-(x_0))\delta_0 \text{ in } B_{r_0}, \quad z_1 = b_0 \text{ on } \partial B_{r_0}, \\ -\Delta z_2 &= -s_- \leq 0 \text{ in } B_{r_0}, \quad z_2 = 0 \text{ on } \partial B_{r_0}. \end{aligned}$$

A direct calculation shows

$$z_1(x) = \frac{m_+(x_0) - m_-(x_0)}{2\pi} \log \frac{1}{|x|} + c_1, \quad (4.15)$$

for some constant c_1 depending only on b_0 and r_0 .

Step 3. We put

$$f = e^{(\alpha_+^* - \varepsilon)z_1} \geq 0, \quad g = e^{(\alpha_+^* - \varepsilon)z_2} \geq 0, \quad p = \frac{\alpha_+^* - \varepsilon}{\alpha_+^* - 2\varepsilon} > 1.$$

Then, (4.15) and (4.11) imply

$$\begin{aligned} \left(\int_{B_{r_0/2}} f^{1/p} \right)^p &= \left(\int_{B_{r_0/2}} e^{(\alpha_+^* - 2\varepsilon)z_1} \right)^p \\ &\geq a \left(\int_{B_{r_0/2}} |x|^{-\frac{(\alpha_+^* - 2\varepsilon)(m_+(x_0) - m_-(x_0))}{2\pi}} \right)^p = +\infty \end{aligned} \quad (4.16)$$

for some $a > 0$. Noting that $z_2 \leq 0$ a.e. in B_{r_0} by the maximum principle, we use (4.10) and Lemma 2.2 to obtain

$$\begin{aligned} \left(\int_{B_{r_0/2}} g^{-\frac{1}{p-1}} \right)^{-(p-1)} &= \left[\int_{B_{r_0/2}} \exp \left(\frac{(\alpha_+^* - 2\varepsilon)(\alpha_+^* - \varepsilon)}{\varepsilon} |z_2| \right) \right]^{-(p-1)} \\ &\geq \left[\int_{B_{r_0/2}} \exp \left(\frac{2\pi}{\|s_-\|_{L^1(B_{r_0})}} |z_2| \right) \right]^{-(p-1)} \geq (2\pi r_0^2)^{-(p-1)}. \end{aligned} \quad (4.17)$$

Step 4. We organize (4.16)-(4.17), the Hölder inequality, (4.14), (4.12), the

Fatou lemma, $0 < r_0 \ll 1$ and the assumption of contradiction, so that

$$\begin{aligned}
+\infty &= \left(\int_{B_{r_0/2}} f^{1/p} \right)^p \left(\int_{B_{r_0/2}} g^{-\frac{1}{p-1}} \right)^{-(p-1)} \\
&\leq \int_{B_{r_0/2}} fg = \int_{B_{r_0/2}} e^{(\alpha_+^* - \varepsilon)z} \leq \int_{B_{r_0/2}} e^{(\alpha_+^* - \varepsilon)v} \leq \liminf_{k \rightarrow \infty} \int_{B_{r_0/2}} e^{(\alpha_+^* - \varepsilon)v_k} \\
&\leq \frac{1}{\mathcal{P}([\alpha_+^* - \varepsilon/2, \alpha_+^*])} \liminf_{k \rightarrow \infty} \int_{[\alpha_+^* - \varepsilon/2, \alpha_+^*]} |B_{r_0/2}|^{\frac{\alpha - (\alpha_+^* - \varepsilon)}{\alpha}} \left(\int_{B_{r_0/2}} e^{\alpha v_k} \right)^{\frac{\alpha_+^* - \varepsilon}{\alpha}} \mathcal{P}(d\alpha) \\
&\leq \frac{1}{\mathcal{P}([\alpha_+^* - \varepsilon/2, \alpha_+^*])} \liminf_{k \rightarrow \infty} \left(1 + \iint_{I \times \Omega} e^{\alpha' v_k} \right) < +\infty,
\end{aligned}$$

a contradiction. \square

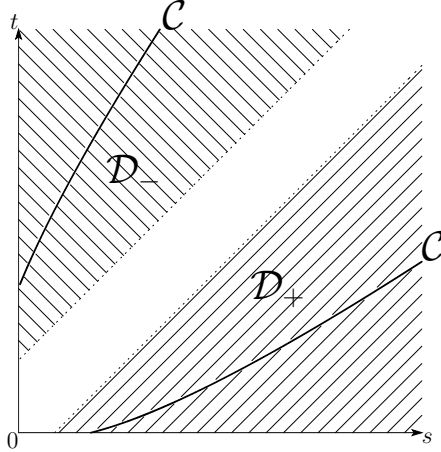
We now arrive at the stage to prove (1.12).

Proof of (1.12) Since v_k is locally uniformly bounded in $\Omega \setminus \mathcal{S}$, and since (1.12) is equivalent to $s_{\pm} = 0$ in (3.3), the proof of (1.12) is reduced to showing (4.9). Moreover, it is reduced to showing (4.8) for $x_0 \in \mathcal{S}$ by virtue of Lemma 4.1.

Property (4.8) is clear for $x_0 \in \mathcal{S}_{\pm} \setminus \mathcal{S}_{\mp}$ by (1.13) and (3.7). For $x_0 \in \mathcal{S}_+ \cap \mathcal{S}_-$, we introduce the sets

$$\begin{aligned}
\mathcal{C} &= \{(s, t) \mid (s - t)^2 = 8\pi(\beta_+ s + \beta_- t), \ s \geq 0, \ t \geq 0\}, \\
\mathcal{D}_+ &= \{(s, t) \mid s - t > 4\pi\beta_+, \ s \geq 0, \ t \geq 0\}, \\
\mathcal{D}_- &= \{(s, t) \mid t - s > 4\pi\beta_-, \ s \geq 0, \ t \geq 0\},
\end{aligned}$$

see the figure below. Then, an elementary calculation shows that $\mathcal{C} \subset \mathcal{D}_+ \cup \mathcal{D}_-$, which implies (4.8) by (1.13). The proof is complete. \square



5 Proof of Theorem 2

Let (λ_k, v_k) be a solution sequence of (1.1) and assume that the alternative (II) in Theorem 1 occurs.

For the purpose, we introduce

$$f_k(t) = \kappa_k \int_I \alpha e^{\alpha t} \mathcal{P}(d\alpha), \quad \kappa_k = \frac{\lambda_k}{\iint_{I \times \Omega} e^{\alpha' v_k}}, \quad F_k(t) = \kappa_k \int_I e^{\alpha t} \mathcal{P}(d\alpha).$$

Then, (1.1) reads

$$-\Delta v_k = f_k(v_k) \text{ in } \Omega \quad v_k = 0 \text{ on } \partial\Omega,$$

and we find the following property by combining (3.5)-(3.6) and (1.12)-(1.13):

$$F_k(v_k) \xrightarrow{*} \sum_{x'_0 \in \mathcal{S}} (\beta_+(x'_0) m_+(x'_0) + \beta_-(x'_0) m_-(x'_0)) \delta_{x'_0} \quad \text{in } \mathcal{M}(\bar{\Omega}), \quad (5.1)$$

where $\beta_{\pm}(x_0)$ is as in (1.10).

We now comply the complex analysis argument developed by [33] to prove Theorem 2.

Proof of Theorem 2 Fix $x_0 \in \mathcal{S}$ and take $\delta > 0$ such that

$$B_{2\delta} \subset \Omega, \quad B_{2\delta} \cap \mathcal{S} = \{x_0\}. \quad (5.2)$$

Define

$$\Gamma = \frac{1}{4\pi} \log(z\bar{z}), \quad I_k = \frac{1}{2} (\partial_z v_k)^2, \quad J_k = \partial_z \Gamma * \{\chi_{B_\delta}(\partial_z F_k(v_k))\}$$

in the usual complex notation $z = X_1 + \iota X_2$, where $\partial_z = \partial/\partial z$ and $*$ denotes the usual convolution. Then, we can easily check that

$$\partial_{\bar{z}} S_k = 0 \text{ in } B_\delta, \quad \text{where } S_k = I_k + J_k \text{ and } \partial_{\bar{z}} = \partial/\partial \bar{z},$$

namely, S_k is a holomorphic function. Hence, there exists S_0 , which is holomorphic in B_δ , such that

$$S_k \rightarrow S_0 \quad \text{locally uniformly in } B_\delta. \quad (5.3)$$

Consider

$$\omega(x) = (m_+(x_0) - m_-(x_0))H(x, x_0) + \sum_{x'_0 \in \mathcal{S} \setminus \{x_0\}} (m_+(x'_0) - m_-(x'_0))G(x, x'_0). \quad (5.4)$$

Note that ω is smooth in B_δ by (5.2). Since

$$v_k \rightarrow v_0 = -(m_+(x_0) - m_-(x_0))\Gamma + \omega \quad \text{in } C_{loc}^2(B_\delta \setminus \{0\})$$

by **(II)-(ii)** in Theorem 1, it holds that

$$I_k \rightarrow I_0 = \frac{(m_+(x_0) - m_-(x_0))^2}{32\pi^2 z^2} - \frac{m_+(x_0) - m_-(x_0)}{4\pi z} \omega_z + \frac{\omega_z^2}{2} \quad (5.5)$$

locally uniformly in $B_\delta \setminus \{0\}$. Moreover, since J_k takes the another form

$$J_k = \partial_{zz} \Gamma * (\chi_{B_\delta} F_k(v_k)) - \partial_z \Gamma * \{\partial_z (\chi_{B_\delta} F_k(v_k))\},$$

(5.1) assures that

$$J_k \rightarrow J_0 = -\frac{\beta_+ m_+(x_0) + \beta_- m_-(x_0)}{4\pi z^2} + J'_0 \quad (5.6)$$

locally uniformly in $B_\delta \setminus \{0\}$, where J'_0 is the non-singular function defined in B_δ .

Organizing (5.3) and (5.5)-(5.6), and comparing the coefficients of the singular parts, we obtain

$$(m_+(x_0) - m_-(x_0))\omega_z(0) = 0, \text{ or } \omega_z(0) = 0 \quad (5.7)$$

by $m_+(x_0) - m_-(x_0) \neq 0$ for $x_0 \in \mathcal{S}$. Finally, (1.14) follows from (5.4) and (5.7). \square

6 Problems on manifolds

In this section, we study the Neri mean field equation on manifolds:

$$\begin{cases} -\Delta v = \lambda \int_I \frac{\alpha \left(e^{\alpha v} - \frac{1}{|\Omega|} \int_\Omega e^{\alpha v} \right)}{\iint_{I \times \Omega} e^{\alpha' v} \mathcal{P}(d\alpha') dx} \mathcal{P}(d\alpha) & \text{on } \Omega \\ \int_\Omega v dx = 0, \end{cases} \quad (6.1)$$

where v is the stream function, $\lambda > 0$ a constant related to the inverse temperature, $\Omega = (\Omega, g)$ a compact and orientable Riemannian surface in dimension two without boundary, g the metric on Ω , $\Delta = \Delta_g$ the Laplace-Beltrami operator, dx the volume element on Ω , $|\Omega|$ the volume of Ω , $\mathcal{P} \in \mathcal{M}(I)$ a Borel probability measure on I , $\mathcal{M}(I)$ the space of measures on I , and $I = [-1, 1]$.

To state the results, we prepare some notations. Let (λ_k, v_k) be a solution sequence of (6.1). Similarly to Section 1, we define the blowup set \mathcal{S} by

$$\mathcal{S} = \mathcal{S}_+ \cup \mathcal{S}_-,$$

$$\mathcal{S}_\pm = \{x_0 \in \Omega \mid \text{there exists } x_k \in \Omega \text{ such that } x_k \rightarrow x_0 \text{ and } v_k(x_k) \rightarrow \pm\infty\},$$

and introduce the measure functions $\nu_{k,\pm} = \nu_{k,\pm}(dx) \in \mathcal{M}(\Omega)$ and $\mu_k = \mu_k(d\alpha dx) \in \mathcal{M}(I \times \Omega)$ defined by

$$\nu_{k,\pm} = \lambda_k \int_{I_\pm} \frac{|\alpha| e^{\alpha v_k}}{\iint_{I \times \Omega} e^{\alpha' v_k} \mathcal{P}(d\alpha)}, \quad (6.2)$$

$$\mu_k(d\alpha dx) = \lambda_k \frac{e^{\alpha v_k}}{\iint_{I \times \Omega} e^{\alpha' v_k} \mathcal{P}(d\alpha)} dx, \quad (6.3)$$

respectively, where $I_+ = [0, 1]$ and $I_- = [-1, 0]$.

With this notation, we review the result of [25].

Proposition 2. *Let (λ_k, v_k) be a solution sequence of (6.1) with $\lambda_k \rightarrow \lambda_0$ for some $\lambda_0 \geq 0$. Then, passing to a subsequence, we have the following alternatives.*

(I') *Compactness:* $\limsup_{k \rightarrow \infty} \|v_k\|_\infty < +\infty$, that is, $\mathcal{S} = \emptyset$.

Then, there exists $v \in E$ such that $v_k \rightarrow v$ in E and v is a solution of (1.1) for $\lambda = \lambda_0$, where

$$E = \{v \in H^1(\Omega) \mid \int_{\Omega} v = 0\}.$$

(II') Concentration: $\limsup_{k \rightarrow \infty} \|v_k\|_{\infty} = +\infty$, that is, $\mathcal{S} \neq \emptyset$.

Then, $\mathcal{S} = \mathcal{S}_+ \cup \mathcal{S}_-$ is finite and the following properties **a')**-**c')** hold:

a') There exists $0 \leq s_{\pm} \in L^1(\Omega) \cap L_{loc}^{\infty}(\Omega \setminus \mathcal{S})$ such that

$$\nu_{k,\pm} \xrightarrow{*} \nu_{\pm} = s_{\pm} + \sum_{x_0 \in \mathcal{S}_{\pm}} m_{\pm}(x_0) \delta_{x_0} \quad \text{in } \mathcal{M}(\Omega), \quad (6.4)$$

with $m_{\pm}(x_0) \geq 4\pi$ for every $x_0 \in \mathcal{S}_{\pm}$, where $\delta_x \in \mathcal{M}(\Omega)$, $x \in \Omega$, denotes the Dirac measure centered at x .

b') There exist $\zeta_{x_0} \in \mathcal{M}(I)$ and $0 \leq r \in L^1(I \times \Omega)$ such that

$$\mu_k \xrightarrow{*} \mu = \mu(d\alpha dx) = r(\alpha, x) \mathcal{P}(d\alpha) dx + \sum_{x_0 \in \mathcal{S}} \zeta_{x_0}(d\alpha) \delta_{x_0}(dx) \quad \text{in } \mathcal{M}(I \times \Omega). \quad (6.5)$$

c') For every $x_0 \in \mathcal{S}$, we have

$$8\pi \int_I \zeta_{x_0}(d\alpha) = \left(\int_I \alpha \zeta_{x_0}(d\alpha) \right)^2 \quad (6.6)$$

$$m_{\pm}(x_0) = \int_{I_{\pm}} |\alpha| \zeta_{x_0}(d\alpha), \quad s_{\pm}(x) = \int_{I_{\pm}} |\alpha| r(\alpha, x) \mathcal{P}(d\alpha), \quad (6.7)$$

where $m_{\pm}(x_0)$ is as in (6.4). Moreover, for every $x_0 \in \mathcal{S}_{\pm} \setminus \mathcal{S}_{\mp}$, it holds that

$$m_{\mp}(x_0) = \int_{I_{\mp}} |\alpha| \zeta_{x_0}(d\alpha) = 0. \quad (6.8)$$

As already stated in Section 1, we shall show the results corresponding to Theorems 1 and 2, except for **(II)-(i)** in Theorem 1. The first result is

Theorem 3. Assume that the alternative **(II')** in Proposition 2 occurs. Then, it holds that $v_k \rightarrow v_0$ in $C_{loc}^2(\Omega \setminus \mathcal{S})$, where

$$v_0(x) = \sum_{x'_0 \in \mathcal{S}} (m_+(x'_0) - m_-(x'_0)) G(x, x'_0) \quad (6.9)$$

with $m_{\pm}(x'_0) \geq 4\pi$ for every $x'_0 \in \mathcal{S}$, and where $G = G(x, y)$ is the Green function defined by

$$-\Delta G(x, y) = \delta_y - \frac{1}{|\Omega|} \quad \text{on } \Omega, \quad \int_{\Omega} G(x, y) dx = 0. \quad (6.10)$$

Moreover, we have

$$(m_+(x_0) - m_-(x_0))^2 = 8\pi(\beta_+(x_0)m_+(x_0) + \beta_-(x_0)m_-(x_0)) \quad (6.11)$$

for every $x_0 \in \mathcal{S}$, where $\beta_{\pm}(x_0)$ is as in (1.10).

To state the second result, we introduce several notations.

Let (λ_k, v_k) be a solution sequence of (6.1). Given $x_0 \in \Omega$, we take an iso-thermal chart (Ψ, U) such that

$$\Phi(x_0) = 0, \quad g = e^{\xi(X)}(dX_1^2 + dX_2^2), \quad X = \Psi(x), \quad \xi(0) = 0. \quad (6.12)$$

Then, $\tilde{v}_k(X) = v_k \circ \Psi^{-1}(X)$ is the solution of

$$-\Delta_X \tilde{v}_k = e^\xi \left(\lambda_k \int_I \frac{\alpha \left(e^{\alpha \tilde{v}_k} - \frac{1}{|\Omega|} \int_\Omega e^{\alpha v_k} \right)}{\iint_{I \times \Omega} e^{\alpha' v_k}} \mathcal{P}(d\alpha) \right) \quad \text{in } \Psi(U). \quad (6.13)$$

Furthermore, we introduce the regular part $H_\Psi = H_\Psi(x, y)$ of the Green function $G = G(x, y)$, defined by (6.10), relative to the isothermal chart satisfying (6.12), namely,

$$H_\Psi(x, y) = G(x, y) - \frac{1}{2\pi} \log \frac{1}{|X - Y|} \quad (6.14)$$

for $(x, y) \in U \times U$, where $X = \Psi(x)$ and $Y = \Psi(y)$.

Under these preparations, the location of the blowup points is characterized in terms of G , H_Ψ and ξ as follows.

Theorem 4. *Let (λ_k, v_k) be a solution sequence of (6.1) and assume that the alternative (II') in Proposition 2 occurs. Then, for every $x_0 \in \mathcal{S}$, we have*

$$\begin{aligned} & \nabla_X \left[H_\Psi(\Psi^{-1}(X), x_0) + \frac{\xi(X)}{8\pi} \right. \\ & \left. + \sum_{x'_0 \in \mathcal{S} \setminus \{x_0\}} \frac{m_+(x'_0) - m_-(x'_0)}{m_+(x_0) - m_-(x_0)} G(\Psi^{-1}(X), x'_0) \right] \Big|_{X=0} = 0. \end{aligned} \quad (6.15)$$

The remainder of this section is devoted to showing Theorems 3 and 4. The proofs are analogue to those of Theorems 1 and 2, and therefore we shall give only a sketch of them here.

Proof of Theorem 3 At first, (6.11) is shown in the same way as (1.13), see Section 4.

Next, we shall give a sketch of the proof of the following property corresponding to Lemma 4.1 in Section 4: for $x_0 \in \mathcal{S}$, if

$$m_+(x_0) - m_-(x_0) > 4\pi\beta_+ \quad \text{or} \quad m_-(x_0) - m_+(x_0) > 4\pi\beta_-$$

then

$$\lim_{k \rightarrow \infty} \iint_{I \times \Omega} e^{\alpha v_k} \mathcal{P}(d\alpha) dx = +\infty. \quad (6.16)$$

Here, we shall only investigate the case that $x_0 \in \mathcal{S}_+ \cap \mathcal{S}_-$ and $m_+(x_0) - m_-(x_0) > 4\pi\beta_+$, since property (6.16) is similarly shown for the other cases.

Fix $x_0 \in \mathcal{S}_+ \cap \mathcal{S}_-$, assume that (6.16) fails, and take an isothermal chart (U, ψ) around x_0 as above. Then, $v_k = v_k(\xi^{-1}(X))$ satisfies

$$-\Delta_X v_k = e^\xi \lambda_k \int_I \frac{\alpha e^{\alpha v_k}}{\iint_{I \times \Omega} e^{\alpha' v_k}} \mathcal{P}(d\alpha) \quad \text{in } \psi(U)$$

and there exists $0 < r_0 \ll 1$ such that $B_{2r_0} \subset \psi(U)$ and $\psi^{-1}(B_{2r_0}) \cap \mathcal{S} = \{x_0\}$. For simplicity, we shall write X and Δ_X by x and Δ , respectively. In the following, we consider the problem on B_{2r_0} , so that

$$-\Delta v_k = e^\varepsilon \lambda_k \int_I \frac{\alpha e^{\alpha v_k}}{\iint_{I \times \Omega} e^{\alpha' v_k}} \mathcal{P}(d\alpha) \quad \text{in } B_{2r_0}.$$

By retaking $0 < r_0 \ll 1$, we obtain $0 < \varepsilon \ll 1$ such that

$$\begin{aligned} \frac{(\alpha_+^* - 2\varepsilon)(\alpha_+^* - \varepsilon)}{\varepsilon} &\leq \frac{2\pi}{\|s_-\|_{L^1(B_{r_0})}}, \quad \alpha_+^* > 2\varepsilon, \\ (\alpha_+^* - 2\varepsilon)(m_+(x_0) - m_-(x_0)) &> 4\pi. \end{aligned}$$

Here, we note that there exist $v \in H_{loc}^1(\Omega \setminus \mathcal{S})$ and $c_0 \in \mathbf{R}$ such that

$$v_k \rightarrow v \quad \text{in } H_{loc}^1(\Omega \setminus \mathcal{S})$$

and

$$-\Delta v = (s_+ - s_-) + \sum_{y_0 \in \mathcal{S}_+} m_+(y_0) \delta_{y_0} - \sum_{y_0 \in \mathcal{S}_-} m_-(y_0) \delta_{y_0} - c_0 \quad \text{on } \Omega$$

with $\int_\Omega v = 0$, see [25].

Let $z = z(x)$ be the very weak solution of

$$\begin{cases} -\Delta z = -s_- + (m_+(x_0) - m_-(x_0))\delta_0 - c_0 & \text{in } B_{r_0} \\ z = b_0 := \min_{\partial B_{r_0}} v & \text{on } \partial B_{r_0}, \end{cases}$$

see [32] for the concept of very weak solutions. Note that $v = v(x)$ satisfies

$$-\Delta v = (s_+ - s_-) + (m_+(x_0) - m_-(x_0))\delta_0 - c_0 \quad \text{in } B_{r_0}.$$

Moreover, we decompose $z = z(x)$ as $z = z_1 + z_2$, where z_1 and z_2 are the solutions of

$$\begin{aligned} -\Delta z_1 &= (m_+(x_0) - m_-(x_0))\delta_0 - c_0 \text{ in } B_{r_0}, \quad z_1 = b_0 \text{ on } \partial B_{r_0}, \\ -\Delta z_2 &= -s_- \leq 0 \text{ in } B_{r_0}, \quad z_2 = 0 \text{ on } \partial B_{r_0}, \end{aligned}$$

respectively. Putting

$$f = e^{(\alpha_+^* - \varepsilon)z_1} \geq 0, \quad g = e^{(\alpha_+^* - \varepsilon)z_2} \geq 0, \quad p = \frac{\alpha_+^* - \varepsilon}{\alpha_+^* - 2\varepsilon} > 1,$$

one can show the following estimates similarly to Section 4:

$$\left(\int_{B_{r_0/2}} f^{1/p} \right)^p = +\infty, \quad \left(\int_{B_{r_0/2}} g^{-\frac{1}{p-1}} \right)^{-(p-1)} \geq (2\pi r_0^2)^{-(p-1)}.$$

Similarly to Section 4 again, we obtain

$$+\infty = \left(\int_{B_{r_0/2}} f^{1/p} \right)^p \left(\int_{B_{r_0/2}} g^{-\frac{1}{p-1}} \right)^{-(p-1)} < +\infty,$$

a contradiction. Hence, (6.16) is established.

Finally, (6.9) is shown by the argument developed in the last part of Section 4, and the proof is complete. \square

Proof of Theorem 4 Let (λ_k, v_k) be a solution sequence of (6.1) and assume that the alternative **(II')** in Proposition 2 occurs.

We put

$$f_k(t) = \kappa_k \int_I \alpha e^{\alpha t} \mathcal{P}(d\alpha) + c_k, \quad \kappa_k = \frac{\lambda_k}{\iint_{I \times \Omega} e^{\alpha' v_k}}, \quad c_k = -\frac{\lambda_k \iint_{I \times \Omega} \alpha e^{\alpha v_k}}{|\Omega| \iint_{I \times \Omega} e^{\alpha v_k}}.$$

Then (6.1) is equivalent to

$$-\Delta v_k = f_k(v_k) \text{ on } \Omega \quad \text{with} \quad \int_{\Omega} v_k = 0.$$

Furthermore, similarly to Section 5, we obtain

$$F_k(v_k) \xrightarrow{*} \sum_{x'_0 \in \mathcal{S}} (\beta_+(x'_0) m_+(x'_0) + \beta_-(x'_0) m_-(x'_0)) \delta_{x'_0} + c_0 v_0 \quad \text{in } \mathcal{M}(\Omega), \quad (6.17)$$

where $\beta_{\pm}(x'_0)$ is as in (1.10), v_0 as in (6.9), $c_0 = \lim_{k \rightarrow \infty} c_k$ and

$$F_k(t) = \kappa_k \int_I e^{\alpha t} \mathcal{P}(d\alpha) + c_k t.$$

To end the proof of Theorem 4, we again comply the complex analysis argument developed by [33].

Fix $x_0 \in \mathcal{S}$ and take an iso-thermal chart satisfying (6.12). We write $\tilde{v}_k(X) = v_k \circ \Psi^{-1}(X)$ by $v_k(X)$ for simplicity. Then (6.13) reads

$$-\Delta_X v_k = e^{\xi} f_k(v_k) \quad \text{in } \Psi(U), \quad (6.18)$$

and there exists $\delta > 0$ such that

$$B_{2\delta} \subset \Psi(U), \quad \Psi^{-1}(B_{2\delta}) \cap \mathcal{S} = \{x_0\}. \quad (6.19)$$

Putting

$$\Gamma = \frac{1}{4\pi} \log(z\bar{z}), \quad I_k = \frac{1}{2} (\partial_z v_k)^2, \quad J_k = \partial_z \Gamma * \{e^{\xi} \chi_{B_{\delta}} (\partial_z F_k(v_k))\},$$

we calculate

$$\partial_{\bar{z}} S_k = 0 \text{ in } B_{\delta}, \quad \text{where } S_k = I_k + J_k \text{ and } \partial_{\bar{z}} = \partial / \partial \bar{z},$$

and hence there exists S_0 , which is holomorphic in B_{δ} , such that

$$S_k \rightarrow S_0 \quad \text{locally uniformly in } B_{\delta}. \quad (6.20)$$

Furthermore, we consider

$$\omega(X) = (m_+(x_0) - m_-(x_0)) H_{\Psi}(x, x_0) + \sum_{x'_0 \in \mathcal{S} \setminus \{x_0\}} (m_+(x'_0) - m_-(x'_0)) G(x, x'_0), \quad (6.21)$$

recall $X = \Psi(x)$, where $H_\Psi = H_\Psi(x, y)$ is the regular part of the Green function defined by (6.14). Note that ω is smooth in B_δ by the smoothness of Ω and (6.19). Since $v_k \rightarrow v_0 = -(m_+(x_0) - m_-(x_0))\Gamma + \omega$ in $C_{loc}^2(B_\delta \setminus \{0\})$ by (6.9), it holds that

$$I_k \rightarrow I_0 = \frac{(m_+(x_0) - m_-(x_0))^2}{32\pi^2 z^2} - \frac{m_+(x_0) - m_-(x_0)}{4\pi z} \omega_z + \frac{\omega_z^2}{2} \quad (6.22)$$

locally uniformly in $B_\delta \setminus \{0\}$. Also, since J_k takes the another form

$$J_k = \partial_{zz}\Gamma * (e^\xi \chi_{B_\delta} F_k(v_k)) - \partial_z \Gamma * \{\partial_z(e^\xi \chi_{B_\delta}) F_k(v_k)\},$$

(6.17) and $\xi(0) = 0$ imply that

$$\begin{aligned} J_k \rightarrow J_0 = & - \frac{\beta_+(x_0)m_+(x_0) + \beta_-(x_0)m_-(x_0)}{4\pi z^2} \\ & - \frac{\beta_+(x_0)m_+(x_0) + \beta_-(x_0)m_-(x_0)}{4\pi z} \xi_z(0) + J'_0 \end{aligned} \quad (6.23)$$

locally uniformly in $B_\delta \setminus \{0\}$, where J'_0 is the non-singular function defined in B_δ .

Organizing (6.20) and (6.22)-(6.23), and comparing the coefficients of the singular parts, we obtain

$$\frac{m_+(x_0) - m_-(x_0)}{\beta_+(x_0)m_+(x_0) + \beta_-(x_0)m_-(x_0)} \omega_z(0) + \xi_z(0) = 0. \quad (6.24)$$

Consequently, (6.15) follows from (6.24), (6.21) and (6.11). \square

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